# Flows on a C*-algebra and cocycle perturbations

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What I call a flow on a C*-algebra $A$ was usually referred to as a **strongly continuous one-parameter automorphism group** of $A$ until some time ago. This topic was extensively studied in 1970's and perhaps in 1980's after the study of (everywhere-defined and so bounded) derivations. At that time the focus was mainly on generators and densely-defined derivations with models from statistical mechanics in mind. A typical question we asked was "Characterize when a densely-defined derivation generates a flow". Another question was related to KMS states (or equilibrium states) asking, e.g., whether they exist uniquely or not. But I suppose this was a bit too vague. The only result worth-mentioning is the uniqueness of KMS states for flows corresponding to the one-dimensional lattice system (or bounded surface energy).

However, clever people soon deserted this field because I think no new results were coming as expected after a general theory (mainly due to Bratteli and Robinson) and some specific results pertaining to AF algebras (mainly due to Sakai) had been established.

See Bratteli-Robinson's book (1979,1981) and Sakai's book (1991) for all these up to around 1980. (Sakai's book is relatively new, but I suppose the main body of the book was written long before.) See also Bratteli's lecture note "Derivations, dissipations and group actions on C*-algebras" (1986) for some progress made after.

## 1 Introduction

By a flow $\alpha$ on a C*-algebra $A$ we mean a continuous homomorphism $\alpha : \mathbf{R} \to \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ is the automorphism group of $A$ equipped with the topology of point-wise convergence.

By an $\alpha$-cocycle $u$ we mean that $u$ is a continuous map from $\mathbf{R}$ into the unitary group of the multiplier algebra $M(A)$ of $A$ with the strict topology such that $u_s\alpha_s(u_t) = u_{s+t},\ s,t \in \mathbf{R}$. If $u$ is an $\alpha$-cocycle then $t \mapsto \mathrm{Ad}\, u_t\alpha_t$ is a flow, called a cocycle perturbation of $\alpha$.

Our far-reaching goal would be classifying the flows up to cocycle perturbations. Since we are not anywhere near this goal, I will first review, as an introduction, two extreme cases, **almost uniformly continuous** flows and **Rohlin** flows. Although Rohlin flows form an interesting subject (and may be the only class of flows susceptible of classification) we will not discuss here; instead focus on flows which are more interesting from a physical point of view. Namely we will discuss the flows which are **approximately inner**, **asymptotically inner**, **quasi-diagonal**, or **pseudo-diagonal**. If the C*-algebra is quasi-diagonal, pseudo-diagonality is the weakest condition among those and implies the existence of KMS states.

We then briefly discuss cocycles; norm-continuous cocycles are describable in a sense and general cocycles can be approximated by norm-continuous one, which is, I think, very much different from the case of von Neumann algebras. We note that the above four conditions are all invariant under cocycle perturbations as expected.

The obvious invariant for cocycle conjugate classes of flows are **crossed products** with dual flows (action), due to Takesaki and Takai. We note that the flows we are interested in have KMS states (if the C*-algebra is unital and finite) and that the traces of the crossed product is described in terms of KMS states (under a mild assumption). The ideal structure of the crossed product could be obtained by studying the ground state (and ceiling state) representations. We will try to describe such crossed products and then conclude the talk by proposing the problem of classifying such crossed products.

## 2 Flows; extreme cases

Let $A$ be a C*-algebra. We denote by $M(A)$ the multiplier algebra of $A$. The strict topology on $M(A)$ is determined by $x \mapsto \|ax\|$ and $x \mapsto \|xa\|$ with $a \in A$. If $A$ is unital then $M(A) = A$ and the strict topology is equivalent to the norm topology.

We call $\alpha$ a **flow** on $A$ if $\alpha$ is a one-parameter automorphism group of $A$ such that $t \mapsto \alpha_t(x)$ is continuous for all $x \in A$.

**Definition 2.1** *We call $\alpha$* **inner** *if there is a unitary flow $u$ in $M(A)$ such that*

$$\alpha_t(x) = \operatorname{Ad} u_t(x) = u_t x u_t^*, \quad x \in A$$

*and $t \mapsto u_t$ is continuous in the strict topology.*

*We call $\alpha$* **universally weakly inner** *if there is a unitary flow $U$ in $A^{**}$ such that $\alpha_t = \operatorname{Ad} U_t|A$ and $t \mapsto U_t$ is continuous in the weak* topology.*

*We call $\alpha$* **uniformly continuous** *if $\|\alpha_t - \mathrm{id}\| \to 0$ as $t \to 0$.*

*We call $\alpha$* **almost uniformly continuous** *if for any $\alpha$-invariant ideal $I$ of $A$ the induced flow on $A/I$ has a non-zero invariant hereditary C*-subalgebra on which it is uniformly continuous. (Then every ideal of $A$ is $\alpha$-invariant.)*

If $\alpha$ is uniformly continuous then the generator

$$\delta_\alpha = \lim_{t \to 0} \frac{\alpha_t - \mathrm{id}}{t}$$

is a bounded operator on $A$ and satisfies

$$\delta_\alpha(x)^* = \delta_\alpha(x^*), \quad x \in A$$

and

$$\delta_\alpha(xy) = x\delta_\alpha(y) + \delta_\alpha(x)y, \quad x, y \in A.$$

A linear operator satisfying these two conditions is called a **derivation** and is automatically bounded. An example of derivation is an **inner** derivation $x \mapsto \mathrm{ad}\, ih(x) = [ih, x]$ with $h \in M(A)_{sa}$. If $A$ is simple then all derivations are inner (Sakai).

**Definition 2.2** *A (non-degenerate) representation $\pi$ of $A$ is* **$\alpha$-covariant** *if there is a unitary flow $U$ on $\mathcal{H}_\pi$ such that $t \mapsto U_t$ is weakly continuous and*

$$\pi\alpha_t(x) = U_t \pi(x) U_t^*, \quad x \in A, \quad t \in \mathbf{R}.$$

There are always covariant representations (since representations of the crossed product give such representations). But it is a non-trivial question to ask whether there is a covariant irreducible (or type II or type III factor) representation (in case $A$ is simple).

**Theorem 2.3** *Consider the following conditions on $\alpha$.*

1. *$\alpha$ is almost uniformly continuous.*

2. *$\alpha$ is universally weakly inner.*

3. *$\alpha^*$ on $A^*$ is strongly continuous, i.e., $\|\phi\alpha_t - \phi\| \to 0$ as $t \to 0$ for $\phi \in A^*$.*

4. *Any irreducible representation of $A$ is $\alpha$-covariant.*

5. *There is a net $(h_\nu)$ in $A_{sa}$ such that $\mathrm{Ad}\, ith_\nu(x) \to \alpha_t(x)$ uniformly in $t$ on every bounded set of $\mathbf{R}$ and for all $x \in A$ and simultaneously $e^{ith_\nu}$ weakly* converges to $U_t$ in $A^{**}$ uniformly in $t$ on every bounded set of $\mathbf{R}$, where $U$ is a unitary flow in $A^{**}$ as in (2).*

6. *$\alpha$ is inner.*

7. *$\alpha$ is uniformly continuous.*

*Then (1)–(5) are equivalent. Moreover if $A$ is simple then (1)–(6) are equivalent. If $A$ is simple and unital all conditions are equivalent.*

Note: (2)$\Leftrightarrow$(5) from Brown-Elliott.

The proofs are not trivial, but the above flows are kind of trivial. One reason for that is they have trivial Borchers spectrum. Let

$$K^1(\mathbf{R}) = \{f \in L^1(\mathbf{R}) \mid \mathrm{supp}(\hat{f}) \text{ is compact}\}.$$

**Definition 2.4** *For $f \in K^1(\mathbf{R})$ and $x \in A$ define $\alpha_f : A \to A$ by*

$$\alpha_f(x) = \int f(t)\alpha_t(x)dt.$$

*For $x \in A$ define the $\alpha$-spectrum of $x$ by*

$$\mathrm{Sp}_\alpha(x) = \text{the kernel of } \{f \in K^1(\mathbf{R}) \mid \alpha_f(x) = 0\}.$$

*For a closed subset of $F$ of $\mathbf{R}$ let*

$$A^\alpha(F) = \{x \in A \mid \mathrm{Sp}_\alpha(x) \subset F\}.$$

*The* **Arveson spectrum** $\mathrm{Sp}(\alpha)$ *of $\alpha$ is the smallest closed subset $F$ satisfying $A^\alpha(F) = A$. Note that $\mathrm{Sp}(\alpha) = -\mathrm{Sp}(\alpha)$.*

**Definition 2.5** *The* **Connes spectrum** $\mathbf{R}_C(\alpha)$ *(resp. the* **Borchers spectrum** $\mathbf{R}_B(\alpha)$*) is*

$$\bigcap_B \mathrm{Sp}(\alpha|B),$$

*where $B$ runs over all the non-zero $\alpha$-invariant hereditary $C^*$-subalgebras of $A$ (resp. those which generate essential ideals of $A$).*

**Remark 2.6** *When $A$ is separable, the Connes spectrum is also given by*

$$\mathbf{R}_C(\alpha) = \bigcap_I \bigcap_u \mathrm{Sp}(\mathrm{Ad}\, u(\alpha \otimes \mathrm{id}) | I \otimes \mathcal{K})$$

*where $I$ runs over the $\alpha$-invariant ideals of $A$ and $u$ runs over the $\alpha \otimes \mathrm{id}$-cocycles in $M(I \otimes \mathcal{K})$.*

*A similar equality holds for the Borchers spectrum $\mathbf{R}_B(\alpha)$ by inserting "essential" in front of ideals.*

**Remark 2.7** *$\mathbf{R}_C(\alpha)$ is a closed subgroup of $\mathbf{R}$, $\mathbf{R}_B(\alpha)$ is a closed subset such that $n\mathbf{R}_B(\alpha) \subset \mathbf{R}_B(\alpha)$ for all $n \in \mathbf{Z}$, and $\mathbf{R}_B(\alpha) \supset \mathbf{R}_C(\alpha)$.*

*Both $\mathbf{R}_C(\alpha)$ and $\mathbf{R}_B(\alpha)$ are invariant under cocycle perturbations.*

*If $A$ is $\alpha$-prime then $\mathbf{R}_C(\alpha) = \mathbf{R}_B(\alpha)$.*

*The crossed product $A \times_\alpha \mathbf{R}$ is prime if and only if $A$ is $\alpha$-prime and $\mathbf{R}_C(\alpha) = \mathbf{R}$. (Olesen-Pedersen)*

**Proposition 2.8** *If $\alpha$ is almost uniformly continuous then*

$$\mathbf{R}_B(\alpha) = \{0\}.$$

*Proof.* If $\alpha$ is uniformly continuous then $\mathrm{Sp}(\alpha)$ is bounded. If $\alpha$ is almost uniformly continuous then one finds an $\alpha$-invariant hereditary C*-subalgebra $B$ of $A$ such that $\alpha|B$ is uniformly continuous and $B$ generates an essential ideal of $A$. Since $\mathbf{R}_B(\alpha) \subset \mathrm{Sp}(\alpha|B)$, one deduces $\mathbf{R}_B(\alpha) = 0$.

We often derive the condition $\mathbf{R}_C(\alpha) = \mathbf{R}$ from a stronger condition:

**Definition 2.9** *We call* $\alpha$ **profound** *if for each* $p \in \mathbf{R}$ *there is a sequence* $(z_n)$ *in* $A$ *such that* $\|z_n\| = 1$,

$$\mathrm{Sp}_\alpha(z_n) \subset (p - 1/n, p + 1/n),$$

$$\|[z_n, x]\| \to 0, \quad x \in A,$$

*and*

$$\|z_n x\| \to 0 \Rightarrow x = 0, \quad x \in A.$$

**Proposition 2.10** *If* $\alpha$ *is profound then* $\mathbf{R}_C(\alpha) = \mathbf{R}$.

*Proof.* Let $B$ be a non-zero $\alpha$-invariant hereditary C*-subalgebra. Let $e \in B_+$ be such that $\mathrm{Sp}_\alpha(e) \subset (-\epsilon, \epsilon)$. Let $(z_n)$ be as in the above definition. Then $ez_ne \neq 0$ for all large $n$. Hence $\mathrm{Sp}(\alpha|B) \cap (p - \epsilon, p + \epsilon) \neq \emptyset$. Thus $\mathrm{Sp}(\alpha|B) = \mathbf{R}$.

Besides the Connes and Borchers spectra of $\alpha$ we also have other similar invariants: the von Neumann algebra versions of the induced flow in an $\alpha$-covariant tracial representation if there is such.

Thus almost uniformly continuous flows locate at one end of the gamut of flows. At the other end there are flows of the following kind:

**Definition 2.11** *We call* $\alpha$ **Rohlin** *if for any finite subset* $\mathcal{F}$ *of* $A$*,* $p \in \mathbf{R}$*, and* $\epsilon > 0$ *there is a unitary* $u \in M(A)$ *such that*

$$\|\alpha_t(u) - e^{ipt}u\| < \epsilon, \quad t \in [-1, 1],$$

*and*

$$\|[u, x]\| < \epsilon, \quad x \in \mathcal{F}$$

This says that the central $\alpha$-cocycle $t \mapsto e^{ipt}$ is trivial, i.e., can be approximated by a sequence of coboundaries $t \mapsto u_n^* \alpha_t(u_n)$ with $(u_n)$ a central sequence of unitaries. This would entail that any $\alpha$-cocycle is trivial, which is a strong property on $\alpha$ we can explore.

**Remark 2.12** *There are Rohlin flows on the Cuntz algebra* $\mathcal{O}_\infty$ *and so on Kirchberg algebras (because* $A \cong A \otimes \mathcal{O}_\infty$ *for such* $A$*).*

*There are Rohlin flows on a unital simple AT algebra* $A$ *of real rank zero if the tracial state space* $T(A)$ *is finite-dimensional and the rank of* $K_1(A)$ *is more than one.*

*There are no Rohlin flows on AF algebras. (If* $A$ *has a unit and has an* $\alpha$*-invariant tracial state then the map* $\mathcal{U}(A) \ni u \mapsto i\tau(u^*\delta_\alpha(u)) \in \mathbf{R}$ *induces a map from* $K_1(A)$ *into* $\mathbf{R}$*. If* $\alpha$ *is a Rohlin flow then its range must be dense.)*

**Remark 2.13** *If* $\alpha$ *is a Rohlin flow then* $\mathbf{R}_C(\alpha) = \mathbf{R}$*. Moreover the strong Connes spectrum is* $\mathbf{R}$ *(or the ideals of the crossed product* $A \times_\alpha \mathbf{R}$ *are all invariant under the dual flow* $\hat{\alpha}$*).*

## 3 Flows in between

**Definition 3.1** *We call* $\alpha$ **approximately inner** *if for any finite subset* $\mathcal{F}$ *of* $A$ *and* $\epsilon > 0$ *there is an* $h \in A_{sa}$ *such that* $\|\alpha_t(x) - \mathrm{Ad}\, e^{ith}(x)\| < \epsilon$ *for* $t \in [-1, 1]$.

*We call* $\alpha$ **asymptotically inner** *(or continuously approximately inner) if there is a continuous function* $h : [0, \infty) \to A_{sa}$ *such that*

$$\alpha_t = \lim_{t\to\infty} \mathrm{Ad}\, e^{ith(s)}(x), \quad x \in A.$$

**Proposition 3.2** (Sakai) *Let* $\alpha$ *be a flow on an AF algebra. Then there is an incresing sequence* $(A_n)$ *of finite-dimensional* $C^*$*-subalgebras of* $A$ *with dense union in* $A$ *such that*

$$D(\delta_\alpha) \supset \bigcup_n A_n,$$

*where* $\delta_\alpha$ *is the generator of* $\alpha$. *Hence there is* $h_n \in A_{sa}$ *such that* $\delta_\alpha | A_n = \mathrm{ad}\, ih_n | A_n$.

It is tempting to conclude that $\mathrm{Ad}\, e^{ith_n} \to \alpha_t$. But this would not follow automatically unless $\bigcup_n A_n$ is dense in the Banach $*$-algebra $D(\delta_\alpha)$ (or $D(\delta_\alpha)$ is approximately finite-dimensional as a Banach algebra). But in general it is not even if $\alpha$ is approximately inner.

If $A$ is separable and $\alpha$ is approximately inner then there is a sequence $(h_n)$ in $A$ such that

$$\operatorname{Ad} e^{ith_n}(x) \to \alpha_t(x)$$

uniformly in $t$ on every compact subset of $\mathbf{R}$. This is equivalent to saying that: the generator $\delta_\alpha$ is the graph limit $L$ of the sequence $\operatorname{ad} ih_n$ of inner derivations.

Here $L$ is defined as follows: $x \in D(L)$ if there is a sequence $(x_n)$ in $A$ such that $x_n \to x$ and $\operatorname{ad} ih_n(x_n)$ converges and then $L(x) = \lim \operatorname{ad} ih_n(x_n)$. However well you may choose the sequence $(h_n)$

$$D = \{x \in D(\delta_\alpha) \mid \lim \operatorname{ad} ih_n(x) = \delta_\alpha(x)\}$$

does not equal $D(\delta_\alpha)$ nor contain all the elements of compact $\alpha$-spectra if $\mathbf{R}_C(\alpha) \neq 0$ and $(A, \alpha)$ has a faithful family of covariant irreducible representations (the latter condition may follow from the approximate innerness but I could not prove).

Maybe because of this we still do not have an intrinsic definition of approximate innerness.

**Remark 3.3** *Apparently asymptotical innerness implies that approximate innerness.*

*All the known examples of approximately inner flows are asymptotically inner (if the $C^*$-algebra is separable).*

**Theorem 3.4** *Let $A$ be a separable $C^*$-algebra. Then $A$ is antiliminary if and only if there is an asymptotically inner flow $\alpha$ such that $\alpha$ is profound.*

*Proof.* There is a sequence $(\pi_n)$ of irreducible representations of $A$ such that $\mathrm{Ran}(\pi_n) \cap \mathcal{K} = \{0\}$ and $\bigcap_n \mathrm{Ker}(\pi_n) = \{0\}$. We construct a flow $\alpha$ such that each $\pi_n$ is covariant under $\alpha$. The construction of $\alpha$ is based on the following lemma.

**Lemma 3.5** *Let $A$ be a separable $C^*$-algebra and let $(a_n)$ be a dense sequence in $A$. Let $(h_n)$ be a sequence in $A_{sa}$ such that*

$$\begin{aligned} \|h_n\| &\le 1, \\ \|[h_n, a_m]\| &\le 2^{-n}\|a_m\|, \quad m \le n, \\ \|[h_n, h_m]\| &\le 2^{-n}, \qquad\quad m < n. \end{aligned}$$

*Let $H_n = \sum_{k=1}^{n} h_k$. Then $\mathrm{Ad}\, e^{itH_n}(x)$ converges as $n \to \infty$ for all $x \in A$ and defines a flow on $A$.*

We choose a unit vector $\eta_n$ from $\mathcal{H}_\pi$. We will construct a central sequence $(v_n)$ in $A$ such that $\pi_k(h_n)\eta_k = 0$ for $k \le n$, $\|\pi_k(v_m)\eta_k\| \approx 1$ for $k \le m$, and

$$\pi_k(h_n v_m)\eta_k \approx 0, \quad m < n, \ k \le n,$$

and

$$\pi_k(h_n v_n)\eta_k \approx \lambda_n \pi_k(v_n)\eta_k, \quad k \le n,$$

where $(\lambda_n)$ is a prescribed dense sequence in $(0, 1)$. This will ensure that $\alpha$ is profound.

**Remark 3.6** *In the above construction we can interpolate linearly between $H_n$ and $H_{n+1}$ to show that $\alpha$ is asymptotically inner. In this way we can construct a flow $\alpha$ on any separable antiliminary $C^*$-algebra $A$ such that $\mathbf{R}_C(\alpha) = \mathbf{R}$. But we do not know if there are infinitely many cocycle conjugacy classes of flows on $A$.*

The condition of asymptotical innerness was introduced to solve the following lifting problem.

**Theorem 3.7** *Let $A$ be a $C^*$-algebra and $I$ an ideal of $A$. Let $B = A/I$ be the quotient of $A$ by $I$ with $Q$ the canonical map of $A$ onto $B$. If $\beta$ is an asymptotically inner flow on $B$ then there is an asymptotically inner flow $\alpha$ on $A$ such that*

$$Q\alpha = \beta Q$$

*and $\alpha|I$ is universally weakly inner.*

A natural question I have not solved yet is: If $\alpha$ is a flow on $A$ such that $\alpha|I$ is asymptotically inner and the induced flow on $B = A/I$ is asymptotically inner, then is $\alpha$ asymptotically inner? The converse certainly holds.

**Example 3.8** *We consider a* **quantum spin system** *over the d-dimensional lattice* $\mathbf{Z}^d$. *We define, as an observable algebra,*

$$A = \bigotimes_{n \in \mathbf{Z}^d} A_n$$

*where* $A_n = M_2$ *(or any matrix algebra). We naturally have the action* $\gamma$ *of* $\mathbf{Z}^d$ *on* $A$ *such that* $\gamma_n(A_m) = A_{m+n}$. *For each finite subset* $\Lambda \subset \mathbf{Z}^d$ *let* $A_\Lambda = \bigotimes_{n \in \Lambda} A_n$ *as a subalgebra of* $A$.

*Let* $\Phi$ *be a function from the finite subsets of* $\mathbf{Z}^d$ *into* $A_{sa}$ *such that* $\Phi(\Lambda) \in A_\Lambda$ *and* $\gamma_n(\Phi(\Lambda)) = \Phi(\Lambda + n)$. *We call* $\Phi$ *an* **interaction**.

*Define for each finite subset* $\Lambda \subset \mathbf{Z}^d$

$$H(\Lambda) = \sum_{X \subset \Lambda} \Phi(X),$$

*which is called a* **local Hamiltonian**.

*Suppose that*

$$\|\Phi\|_\lambda = \sum_{k=0}^{\infty} e^{\lambda k} \sum_{X \ni 0,\, |X| = k+1} \|\Phi(X)\| < \infty$$

*for some* $\lambda > 0$. *(In particular more restrictively suppose that* $\Phi$ *is of finite range, i.e.,* $\Phi(X) = 0$ *whenever the diameter of* $X$ *is greater than some constant.) Then a flow* $\alpha^\Phi$ *on* $A$ *can be defined by the limit*

$$\operatorname{Ad} e^{itH(\Lambda)}(x) \to \alpha_t^\Phi(x)$$

*as* $\Lambda \uparrow \mathbf{Z}^d$. *The flow* $\alpha^\Phi$ *is asymptotically inner.*

*We will refer to this type of flows as* **quantum spin flows**.

Let $T$ be a bounded operator on a Hilbert space $\mathcal{H}$. $T$ is called **quasi-diagonal** if there is an increasing sequence $(E_n)$ of finite-rank projections on $\mathcal{H}$ such that

$$E_n \to 1,$$

and

$$\|[E_n, T]\| \to 0.$$

If $T$ is self-adjoint then $T$ is quasi-diagonal. If $T$ is an unbounded self-adjoint operator we can still say that $T$ is quasi-diagonal (due to the Weyl-von Neumann theorem).

This notion can be extended to a set of bounded operators.

When $A$ is a C*-algebra, $A$ is called **quasi-diagonal** if there is a faithful representation $\pi$ of $A$ such that $\pi(A)$ is quasi-diagonal. Easy examples include AF algebras and commutative C*-algebras.

We extend this notion to flows in two ways.

**Definition 3.9** *Given a Hilbert space $\mathcal{H}$, let $A$ be a norm-closed $^*$-algebra of bounded operators on $\mathcal{H}$ and let $U$ be a unitary flow on $\mathcal{H}$ such that $U_t x U_t^* \in A$ for $t \in \mathbf{R}$ and $t \mapsto U_t x U_t^*$ is norm-continuous for any $x \in A$.*

*We call $(A, U)$ to be* **quasi-diagonal** *if for any finite set $\mathcal{F}$ of $A$, any finite set $\omega$ of $\mathcal{H}$, and $\epsilon > 0$ there is a finite-rank projection $E$ on $\mathcal{H}$ such that*

$$\|[E, x]\| \leq \epsilon \|x\|, \quad x \in \mathcal{F},$$

$$\|(1 - E)\xi\| \leq \epsilon \|\xi\|, \quad \xi \in \omega,$$

*and*

$$\|[E, U_t]\| < \epsilon, \quad t \in [-1, 1].$$

*We call $(A, U)$ to be* **pseudo-diagonal** *if for any finite set $\mathcal{F}$ of $A$, any finite set $\omega$ of $\mathcal{H}$, and $\epsilon > 0$ there is a finite-rank projection $E$ on $\mathcal{H}$ and a unitary flow $V$ on $E\mathcal{H}$ such that*

$$\|[E, x]\| \leq \epsilon \|x\|, \quad x \in \mathcal{F},$$

$$\|(1 - E)\xi\| \leq \epsilon \|\xi\|, \quad \xi \in \omega,$$

*and*

$$\|EU_t x U_t^* E - V_t E x E V_t^*\| \leq \epsilon \|x\|, \quad x \in \mathcal{F}, \ t \in [-1, 1].$$

*Let $A$ be a $C^*$-algebra and let $\alpha$ be a flow on $A$. We call $\alpha$ to be* **quasi-diagonal** *(resp.* **pseudo-diagonal***) if $(A, \alpha)$ has a covariant representation $(\pi, U)$ on a Hilbert space $\mathcal{H}_\pi$, with $\pi$ faithful and non-degenerate, such that $(\pi(A), U)$ is quasi-diagonal (resp. pseudo-diagonal).*

Note that $\alpha$ being quasi-diagonal or pseudo-diagonal is much stronger than $A \times_\alpha \mathbf{R}$ being diagonal.

The condition $\|[E, U_t]\| < \epsilon,\ t \in [-1, 1]$ in the definition of quasi-diagonality can be replaced by

$$\|[E, H]\| < \epsilon,$$

where $H$ is the self-adjoint generator of $U$: $U_t = e^{itH}$.

**Remark 3.10** *If $\alpha$ is quasi-diagonal (resp. pseudo-diagonal) and $B$ is an $\alpha$-invariant $C^*$-subalgebra of $A$, then $\alpha|B$ is quasi-diagonal (resp. pseudo-diagonal).*

*This kind of property is not at all clear for approximately inner flows.*

The following three theorems can be proved by adopting Voiculescu's arguments to the present situation.

**Theorem 3.11** *Let $\alpha$ be a flow on a $C^*$-algebra $A$. Then the following conditions are equivalent:*

1. *$\alpha$ is quasi-diagonal.*

2. *For any finite subset $\mathcal{F}$ of $A$ and $\epsilon > 0$ there is a finite-dimensional $C^*$-algebra $B$, a flow $\beta$ on $B$, and a CP map $\phi$ of $A$ into $B$ such that*

$$\|\phi\| \leq 1, \quad \|\phi(x)\| \geq (1-\epsilon)\|x\|,$$

$$\|\phi(x)\phi(y) - \phi(xy)\| \leq \epsilon\|x\|\|y\|, \quad x, y \in \mathcal{F},$$

   *and*

$$\|\beta_t\phi - \phi\alpha_t\| < \epsilon, \quad t \in [-1, 1].$$

3. *For any finite subset $\mathcal{F}$ of $A$ and $\epsilon > 0$ there is a covariant representation $(\pi, U)$ as well as a finite-rank projection $E$ on $\mathcal{H}_\pi$ such that*

$$\|E\pi(x)E\| \geq \|x\| - \epsilon,$$

$$\|[E, \pi(x)]\| \leq \epsilon\|x\|, \quad x \in \mathcal{F},$$

   *and*

$$\|[E, U_t]\| < \epsilon, \quad t \in [-1, 1].$$

**Theorem 3.12** *Let $\alpha$ be a flow on a $C^*$-algebra $A$. Then the following conditions are equivalent:*

1. *$\alpha$ is pseudo-diagonal.*

2. *For any finite subset $\mathcal{F}$ of $A$ and $\epsilon > 0$ there is a finite-dimensional $C^*$-algebra $B$, a flow $\beta$ on $B$, and a CP map $\phi$ of $A$ into $B$ such that*

$$\|\phi\| \leq 1, \quad \|\phi(x)\| \geq (1-\epsilon)\|x\|,$$

$$\|\phi(x)\phi(y) - \phi(xy)\| \leq \epsilon\|x\|\|y\|, \quad x, y \in \mathcal{F},$$

*and*

$$\|\beta_t\phi(x) - \phi\alpha_t(x)\| \leq \epsilon\|x\|, \quad x \in \mathcal{F}, \ t \in [-1, 1].$$

3. *For any finite subset $\mathcal{F}$ of $A$ and $\epsilon > 0$ there is a covariant representation $(\pi, U)$, a finite-rank projection $E$ on $\mathcal{H}_\pi$, and a unitary flow $V$ on $E\mathcal{H}_\pi$ such that*

$$\|E\pi(x)E\| \geq (1-\epsilon)\|x\|,$$

$$\|[E, \pi(x)]\| \leq \epsilon\|x\|, \quad x \in \mathcal{F},$$

*and*

$$\|EU_t\pi(x)U_t^*E - V_tE\pi(x)EV_t^*\| \leq \epsilon\|x\|, \ x \in \mathcal{F}, \ t \in [-1, 1].$$

**Theorem 3.13** *Let $\alpha$ be a quasi-diagonal (resp. pseudo-diagonal) flow on $A$. Then for any covariant representation $(\rho, V)$ of $A$ such that $\rho \times V$ is a faithful representation of $A \times_\alpha \mathbf{R}$ and* $\mathrm{Ran}(\rho \times V) \cap \mathcal{K}(\mathcal{H}_\rho) = \{0\}$, *$(\rho(A), V)$ is quasi-diagonal (resp. pseudo-diagonal).*

This follows by slightly modifying the proof of Voiculescu's Weyl-von Neumann theorem. It is not too difficult to handle one unbounded self-adjoint operator associated with $A \times_\alpha \mathbf{R}$ in addition to itself.

**Definition 3.14** *Let $A$ be a UHF algebra and $\alpha$ a flow $\alpha$ on $A$. We call $\alpha$ a* **UHF flow** *if there is a sequence $(k_n)$ of integers such that $k_n \geq 2$ and*

$$A = \bigotimes_{n=1}^{\infty} M_{k_n}$$

*and*

$$\alpha_t = \bigotimes \operatorname{Ad} e^{ith_n},$$

*where $h_n \in (M_{k_n})_{sa}$.*

Going back to the quantum spin flows, if the interaction $\Phi$ satisfies that $\Phi(X) = 0$ whenever $|X| > 1$ then $\alpha^{\Phi}$ is a UHF flow.

**Definition 3.15** *Let $A$ be an AF algebra and $\alpha$ a flow on $A$. We call $\alpha$ an* **AF flow** *if there is an increasing sequence $(A_n)$ of finite-dimensional $C^*$-subalgebras of $A$ with dense union such that*

$$\alpha_t(A_n) = A_n.$$

*We call $\alpha$ an* **approximate AF flow** *if there is an increasing sequence $(A_n)$ of finite-dimensional $C^*$-subalgebras of $A$ with dense union such that*

$$\sup_{t\in[0,1]} \operatorname{dist}(A_n, \alpha_t(A_n)) \to 0$$

*as $n \to \infty$*

When $B$ and $C$ are subsets of $A$ and $\delta > 0$ we write $B \overset{\delta}{\subset} C$ if for any $x \in B$ there is $y \in C$ such that $\|x - y\| \leq \delta \|x\|$. The distance of $B$ and $C$ is defined by

$$\operatorname{dist}(B, C) = \inf\{\delta > 0 \mid B \overset{\delta}{\subset} C,\ C \overset{\delta}{\subset} B\}.$$

**Proposition 3.16** *Let $\alpha$ be a flow on an AF algebra. Then $\alpha$ is an approximate AF flow if and only if it is a cocycle perturbation of an AF flow.*

The "if" part is almost obvious. The difficult part is the "only if"; the proof I have is rather roundabout.

In the case of quantum spin flows if the interaction is of finite range and

$$\Phi(\Lambda) \in \bigotimes_{n \in \Lambda} D_n = A_\Lambda \cap D$$

where $D_n$ is the diagonal matrices of $A_n = M_2$ and $D$ is the C*-subalgebra generated by all $D_n$, then $\alpha^\Phi$ is an AF flow.

(Suppose that $\Phi(X) = 0$ if the diameter of $X$ is greater than $K > 0$. Let $\Lambda$ be a finite subset of $\mathbf{Z}^d$. The C*-subalgebra generated by $A_\Lambda$ and $D_n$ with $n$ within the $K$-neighborhood of $\Lambda$ is left invariant under $\alpha^\Phi$.)

**Remark 3.17** *The AF flows already form a rich class of flows. We do not know to this day if there is a quantum spin flow which is not an approximate AF flow.*

**Proposition 3.18** *If $\alpha$ is an AF flow then $\alpha$ is quasi-diagonal.*

*Proof.* We choose a maximal abelian C*-subalgebra $D_n$ of $A_n \cap A'_{n-1}$ such that $\alpha_t|D_n = \mathrm{id}$ and let $D$ be the C*-subalgebra generated by all $D_n$, which is a maximal abelian C*-subalgebra of $A$. Let $\phi$ be a character of $D$ which extends to a pure $\alpha$-invariant state of $A$.

Let $(\pi_\phi, U^\phi)$ be the GNS representation;

$$U_t^\phi \pi_\phi(x)\Omega_\phi = \pi_\phi \alpha_t(x)\Omega_\phi, \quad x \in A.$$

Let $E_n$ be the finite-rank projection onto $\pi_\phi(A_n)\Omega_\phi$. Then

$$[E_n, \pi_\phi(x)] = 0, \quad x \in A_n,$$

and

$$[E_n, U_t^\phi] = 0.$$

In the case of quantum spin flows if $\Phi(X) \in \bigotimes_{n \in X} D_n$ without the condition of finite range, we still have the above conclusion. In this case $\alpha^\Phi$ may not be an AF flow; but of more general kind of AF flow.

**Proposition 3.19** *If $\alpha$ is an approximately inner flow on a quasi-diagonal $C^*$-algebra then $\alpha$ is pseudo-diagonal.*

*Proof.* We suppose that $A$ acts non-degenerately on a Hilbert space $\mathcal{H}$ such that $A$ is a quasi-diagonal set of $\mathcal{B}(\mathcal{H})$.

Let $\mathcal{F}$ be a finite subset of $A$ and $\epsilon > 0$. By the assumption there is an $h = h^* \in A$ such that $\|\alpha_t(x) - \mathrm{Ad}\, e^{ith}(x)\| \le \epsilon/3\|x\|$ for $x \in \mathcal{F}$ and $t \in [-1,1]$. There is a finite-rank projection $E$ on $\mathcal{H}$ such that $\|ExE\| \ge (1-\epsilon)\|x\|$ and $\|[E,x]\| \le \epsilon\|x\|$ for $x \in \mathcal{F}$, and $\|[E,h]\| < \epsilon/3$. Since $\|Ee^{ith}E - e^{itEhE}E\| < \epsilon/3$ for $t \in [-1,1]$, it follows that

$$\|E\alpha_t(x)E - \mathrm{Ad}\, e^{itEhE}(ExE)\| \le \epsilon\|x\|, \quad x \in \mathcal{F}.$$

Note also that $\|ExEyE - ExyE\| \le \epsilon\|x\|\|y\|$ for $x, y \in \mathcal{F}$. By setting $B = \mathcal{B}(E\mathcal{H})$, $\beta_t = \mathrm{Ad}\, e^{itEhE}$, and $\phi(x) = ExE$, we obtain the desired objects for $(\mathcal{F}, \epsilon)$.

**Proposition 3.20** *Let $A$ denote the gauge-invariant CAR algebra. Then any flow on $A$ is quasi-diagonal.*

*Proof.* There is a decreasing sequence $(I_n)$ of ideals in $A$ such that $A/I_1 \cong \mathbf{C}$, $I_{n-1}/I_n \cong \mathcal{K}$ for $n\,1$, and $\bigcap_n I_n = \{0\}$.

**Proposition 3.21** *Let $\Omega$ be a compact Hausdorff space and $\alpha$ a flow of homeomorphisms of $\Omega$. If $\alpha$ has no fixed points then the induced flow on $C(\Omega)$ is not pseudo-diagonal.*

*Proof.* The proof uses the existence of KMS states which follow from pseudo-diagonality.

**Proposition 3.22** *Let $\mathbf{D}$ denote the unit disk $\{z \in \mathbf{C} \mid |z| \leq 1\}$ and define a flow $\alpha$ by $\alpha_t(z) = e^{it}z$. Then the induced flow on $C(\mathbf{D})$ is quasi-diagonal.*

When the C*-algebra is quasi-diagonal the relations among the four notions are

$$\begin{array}{ccc} \text{Asymptotically inner} & \Rightarrow & \text{Approximately inner} \\ & & \Downarrow \\ \text{Quasi}-\text{diagonal} & \Rightarrow & \text{Pseudo}-\text{diagonal} \end{array}$$

## 4 Cocycles

If $\alpha$ is a flow on $A$, then $\alpha$ extends to a one-parameter automorphism group of $M(A)$ such that $t \mapsto \alpha_t(x)$ is continuous in the strict topology for $x \in M(A)$. We denote such an extension by the same symbol $\alpha$.

**Definition 4.1** *Let $\alpha$ be a flow on a $C^*$-algebra $A$. We call $u$ an* $\alpha$-**cocycle** *(in $M(A)$) if $u$ is a continuous function of* $\mathbf{R}$ *into the unitary group of $M(A)$ such that $u_s\alpha_s(u_t) = u_{s+t},\ s,t \in$* $\mathbf{R}$. *Moreover if $u_t \in A + \mathbf{C}1$ then we call $u$ an $\alpha$-cocycle in $A$.*

Let $w$ be a unitary. Then $t \mapsto w\alpha_t(w^*)$ is an $\alpha$-cocycle, called a coboundary. More generally if $u$ is an $\alpha$-cocycle and $w$ is a unitary, then

$$t \mapsto wu_t\alpha_t(w^*)$$

is an $\alpha$-cocycle.

Let $h \in A_{sa}$ and define

$$u_t = \sum_{n=0}^{\infty} i^n \int_{\Omega_n} \alpha_{t_1}(h)\alpha_{t_2}(h)\cdots\alpha_{t_n}(h)dt_1\cdots dt_n,$$

where if $t \geq 0$

$$\Omega_n = \{(t_1, \ldots, t_n) \mid 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq t\}$$

and if $t \leq 0$ similarly. Then $u_t$ is differentiable and satisfies $du_t/dt = u_t i\alpha_t(h)$. Then one deduces that $u$ is an $\alpha$-cocyle in $A$.

If $u$ is an $\alpha$-cocycle then we denote by $\mathrm{Ad}u\alpha$ the flow $t \mapsto \mathrm{Ad}u_t\alpha$ on $A$. If $u$ is differentiable and $ih = du_t/dt|_{t=0}$ then $\mathrm{Ad}\,u\alpha$ is generated by $\delta_\alpha + \mathrm{ad}\,ih$.

**Proposition 4.2** *Suppose that $A$ is unital and let $u$ be an $\alpha$-cocycle. Then for any $\epsilon > 0$ there is an analytic cocycle $v$ and a unitary $w$ such that $\|w - 1\| < \epsilon$ and*

$$u_t = wv_t\alpha_t(w^*).$$

*Proof.* We define a flow $\gamma$ on $A \otimes M_2$ by

$$\gamma_t \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} \alpha_t(x_{11}) & \alpha_t(x_{12})u_t^* \\ u_t\alpha_t(x_{21}) & u_t\alpha_t(x_{22})u_t^* \end{pmatrix}.$$

Note that $\gamma_t(e_{21}) = u_t e_{21}$. There is a $\gamma$-analytic element $x$ such that $x = e_{22}xe_{11}$ and $\|x - e_{21}\| \approx 0$. We may replace $x$ by $x(x^*x)^{-1}$. Let $x = w \otimes e_{21}$ where $w \in U(A)$. Then $t \mapsto u_t\alpha_t(w)$ is analytic. Thus $v_t = w^*u_t\alpha_t(w)$ is an analytic $\alpha$-cocycle.

**Proposition 4.3** *Suppose that $A$ is unital and let $u$ be an $\alpha$-cocycle. Then for any $\epsilon > 0$ there is an entire non-unitary cocycle $v$ and an invertible element $w$ such that $\|w - 1\| < \epsilon$ and*

$$u_t = wv_t\alpha_t(w^{-1}).$$

*Proof.* In the above proof if we drop the condition that $w \in U(A)$ then we can assume that $t \mapsto \gamma_t(x) = u_t\alpha_t(w) \otimes e_{21}$ is entire for $x = w \otimes e_{21}$. Then $v_t = w^{-1}u_t\alpha_t(w)$ satisfies

$$v_s\alpha_s(v_t) = w^{-1}u_s\alpha_s(w)\alpha_s(w^{-1}u_t\alpha_t(w)) = v_{s+t}.$$

**Theorem 4.4** *Let $u$ be an $\alpha$-cocycle in $M(A)$, $p \in A$, and $\epsilon > 0$. Then there is an $\alpha$-cocycle $v$ in $A$ such that*

$$\|(u_t - v_t)p\| < \epsilon, \ t \in [-1, 1].$$

*Proof.* If there is an $e \in A_{sa}$ such that $ep \approx p$, $eu_tp \approx u_tp$, $\delta_\alpha(e) \approx 0$, and $t \mapsto eu_te$ is differentiable then we set

$$d(eu_te)/dt|_{t=0} = ih.$$

Since $h^* = h$ we define an $\alpha$-cocycle $v$ by

$$dv_t/dt = v_t\alpha_t(ih)$$

with $v_0 = 1$. Then it would follow that $u_tp \approx v_tp$. The main problem is to find such an $e$ (for an $\alpha$-cocycle close to $u$).

**Proposition 4.5** *The four properties, approximate innerness, asymptotic innerness, pseudo-diagonality, quasi-diagonality, are invariant under cocycle perturbation.*

**Proposition 4.6** *Let $B$ be an $\alpha$-invariant hereditary $C^*$-subalgebra of $A$. Then the following hold:*

1. *If $\alpha$ is approximately inner then $\alpha|B$ is approximately inner.*

2. *If $B$ generates $A$ as an ideal then the converse holds.*

*The above statements holds for pseudo-diagonality and quasi-diagonality (instead of approximate innerness).*

*If each of $A$ and $B$ have a strictly positive element, the above statements hold for asymptotical innerness (instead of approximate innerness).*

## 5 KMS states

Let $A$ be a unital C*-algebra and $\alpha$ a flow on $A$.

**Definition 5.1** *Let $c \in \mathbf{R}$. A state $\omega$ on $A$ is called an* $\alpha$-**KMS state** *at $c$ if $\omega\alpha_t = \omega$ for $t \in \mathbf{R}$ and $\omega(xy) = \omega(y\alpha_{ic}(x))$ for all $\alpha$-entire $x, y \in A$.*

In the above definition if $c \neq 0$ then the invariance $\omega\alpha_t = \omega$ follows from the other part of the condition. If $c = 0$ then the KMS state is an $\alpha$-invariant tracial state.

**Proposition 5.2** *A state $\omega$ on $A$ is a KMS state at $c > 0$ if and only if for any $x, y \in A$ there is a bounded continuous function $f$ on $\mathbf{C}_c = \{z \in \mathbf{C} \mid 0 \leq \Im(z) \leq c\}$ such that $f$ is holomorphic in the interior of $\mathbf{C}_c$ and*

$$f(t) = \omega(y\alpha_t(x)), \quad t \in \mathbf{R},$$

*and*

$$f(t + ic) = \omega(\alpha_t(x)y), \quad t \in \mathbf{R}.$$

**Definition 5.3** *A state $\omega$ is called a $\alpha$-**ground state** (resp. $\alpha$-**ceiling state**) if*

$$-i\omega(x^*\delta_\alpha(x)) \geq 0 \quad (\text{resp.} \leq 0)$$

*for $x \in D(\delta_\alpha)$.*

In the above the invariance $\omega\alpha_t = \omega$ follows automatically. (If $x = x^* \in D(\delta_\alpha)$ then $\omega\delta_\alpha(x^2) = \omega(\delta_\alpha(x)x) + \omega(x\delta_\alpha(x)) = 0$.)

If $\omega$ is a ground state then by defining a unitary flow $U$ on the GNS representation space associated with $\omega$ by

$$U_t\pi_\omega(x)\Omega_\omega = \pi_\omega\alpha_t(x)\Omega_\omega, \quad x \in A,$$

we derive that $H \geq 0$, where $H$ is the self-adjoint generator of $U$, from $-i\omega(x^*\delta_\alpha(x)) = \langle \pi_\omega(x)\Omega_\omega, H\pi_\omega(x)\Omega_\omega \rangle$.

**Proposition 5.4** *A state $\omega$ is a ground state if and only if for any $x, y \in A$ there is a bounded continuous function $f$ on $\mathbf{C}_\infty = \{z \in \mathbf{C} \mid \Im(z) \geq 0\}$ such that $f$ is holomorphic in the interior of $\mathbf{C}_\infty$ and*

$$f(t) = \omega(x\alpha_t(y)), \quad t \in \mathbf{R}.$$

If $A = M_n$ then any flow $\alpha$ on $A$ is given as $\alpha_t = \mathrm{Ad}\, e^{ith}$ for some $h \in A_{sa}$. For $c \in \mathbf{R}$ define a state $\omega_c$ on $A$ by

$$\omega_c(x) = \mathrm{Tr}(xe^{-ch})/\mathrm{Tr}(e^{-ch}), \quad x \in A,$$

where Tr is the trace on $A = M_n$.

**Then $\omega_c$ is a unique $\alpha$-KMS state at $c$.** This follows by computation:

$$\omega_c(y\alpha_{ic}(x)) = C\mathrm{Tr}(ye^{-ch}xe^{ch}e^{-ch}) = C\mathrm{Tr}(xye^{-ch}) = \omega_c(xy)$$

where $C = \mathrm{Tr}(e^{-ch})^{-1}$. If $\omega$ is a KMS state, then letting $\rho \in A$ with $\omega(\cdot) = \mathrm{Tr}(\rho\cdot)$ we compute:

$$\mathrm{Tr}(\rho xy) = \mathrm{Tr}(\rho ye^{-ch}xe^{ch}) = \mathrm{Tr}(e^{-ch}xe^{ch}\rho y), \quad x, y \in A,$$

which entails $e^{ch}\rho = c1$ for some $c > 0$.

**Proposition 5.5** *If $\alpha$ is a UHF flow then it has a unique KMS state for all $c \in \mathbf{R}$.*

*Proof.* If $\alpha_t = \bigotimes \operatorname{Ad} e^{ith_n}$ on $A = \bigotimes M_{k_n}$, then the KMS state $\omega$ at $c$ is obtained as the infinite tensor product

$$\bigotimes_n \operatorname{Tr}(\,\cdot\, e^{-ch_n})/\operatorname{Tr}(e^{-ch_n}).$$

Let $\alpha$ be an AF flow on a unital AF algebra $A$ and let $(A_n)$ be an increasing sequence of finite-dimensional C*-subalgebras of $A$ with dense union such that $\alpha_t(A_n) = A_n$. Let $Z_n = A_n \cap A_n'$ and $Z_n \cong \mathbf{C}^{k_n}$. Let $\omega$ be a KMS state of $A$. Then $\omega|A_n$ is determined by $\omega|Z_n$ which corresponds to a point in the $k_n - 1$ simplex $\Delta_n$. Denoting the map $\Delta_{n+1} \to \Delta_n$ by $S_n$ (giving $\phi|Z_{n+1} \mapsto \phi|Z_n$ with $\phi$ a KMS state on $A_{n+1}$) we conclude that the set of KMS states of $A$ is given as the projective limit of

$$\Delta_1 \overset{S_1}{\leftarrow} \Delta_2 \overset{S_2}{\leftarrow} \Delta_3 \leftarrow \cdots .$$

Thus the KMS states of an AF flow are describable in a sense.

**Remark 5.6** *If $\alpha$ is a Rohlin flow on a unital C*-algebra, then $\alpha$ has no KMS states at non-zero c. (If u is a unitary such that $\alpha_t(u) \approx e^{itp}u$ and $\omega$ is a KMS state at c, then $1 = \omega(uu^*) = \omega(u^*\alpha_{ic}(u)) \approx e^{-pc}$.)*

**Proposition 5.7** *Suppose that $\alpha$ is a pseudo-diagonal flow on a unital $C^*$-algebra $A$. Then $\alpha$ has a KMS state for all inverse temperatures including $\pm\infty$.*

*Proof.* Let $\mathcal{F}$ be a finite subset $\mathcal{F}$ of $A$ and $\epsilon > 0$. For each $(\mathcal{F}, \epsilon)$ we have a flow $\beta$ on a finite-dimensional C*-algebra $B$ and a CP map $\phi$ of $A$ into $B$ such that

$$\phi(1) = 1, \quad \|\phi(x)\| \geq (1-\epsilon)\|x\|$$
$$\|\phi(x)\phi(y) - \phi(xy)\| \leq \epsilon\|x\|\|y\|, \quad x, y \in \mathcal{F}$$

and

$$\|\beta_t\phi(x) - \phi\alpha_t(x)\| \leq \epsilon\|x\|, \quad x \in \mathcal{F}, \ t \in [-1, 1].$$

Here we have replaced the condition $\|\phi\| \leq 1$ by $\phi(1) = 1$ since $A$ is unital.

There is a self-adjoint $h \in B$ such that $\beta_t = \mathrm{Ad}\, e^{ith}$. We fix $\gamma \in \mathbf{R}$ and define a state $\varphi$ on $B$ by

$$\varphi(Q) = \mathrm{Tr}(e^{-\gamma h}Q)/\mathrm{Tr}(e^{-\gamma h}),$$

where Tr is a trace on $B$. Then we know that $\varphi$ is a KMS state on $B$ with respect to $\beta$ at inverse temperature $\gamma$.

We set a state $f_{(\mathcal{F},\epsilon)}$ on $A$ by $\varphi\phi$, where $\varphi$ and $\phi$ depend on $(\mathcal{F}, \epsilon)$. Let $f$ be a weak*-limit point of $f_{(\mathcal{F},\epsilon)}$, where the set $X$ of $(\mathcal{F}, \epsilon)$ is a directed set in an obvious way. We fix a Banach limit $\psi$ on $L^\infty(X)$ such that $f(x)$ is the $\psi$ limit of $(\mathcal{F}, \epsilon) \mapsto f_{(\mathcal{F},\epsilon)}(x)$ for $x \in A$. Note that $f(x\alpha_t(y))$ is the $\psi$ limit of $(\mathcal{F}, \epsilon) \mapsto \varphi(\phi(x\alpha_t(y)))$, which is close to $\varphi(\phi(x)\beta_t\phi(y))$ around $\infty$. Thus one can conclude that $f$ is a KMS state at $\gamma$.

A similar proof works for a KMS state for $\gamma = \pm\infty$ (or a ground state and ceiling state).

**Lemma 5.8** *Let $u$ be an $\alpha$-cocycle and express $u$ as $u_t = wv_t\alpha_t(w^{-1})$ where $v$ is an entire non-unitary $\alpha$-cocycle. For a state $\omega$ of $A$ and $c \in \mathbf{R}$ define a state $\omega'$ on $A$ by*

$$\omega'(a) = \frac{\omega(w^{-1}awv_{ic})}{\omega(v_{ic})}.$$

*If $\omega$ is a KMS state at $c$ with respect to $\alpha$ then $\omega'$ is a KMS state at $c$ with respect to $\mathrm{Ad}\, u\alpha$.*

*Proof.* Note $\omega(v_{ic}) > 0$. This follows formally since

$$\omega(v_{ic}) = \omega(w^{-1}wv_{ic}) = \omega(wv_{ic}\alpha_{ic}(w)) = \omega(u_{ic}),$$

which is positive because $t \mapsto \omega(u_t)$ is positive-definite. This is because

$$\omega(u_{t_i - t_j}) = \omega(u_{t_i}\alpha_{t_i}(u_{-t_j})) = \omega(\alpha_{-t_i}(u_{t_i})\alpha_{-t_j}(u^*_{t_j})).$$

The numerator for $a = x^*x$ is non-negative because

$$\omega(w^{-1}x^*xwv_{ic}) = \omega(x^*xu_{ic}) = \omega(xu_{ic}\alpha_{ic}(x^*))$$

and $t \mapsto \omega(xu_t\alpha_t(x^*))$ is positive-definite.

Let $\alpha' = \mathrm{Ad}\, u\alpha$. Formally $\omega(v_{ic})\omega'(xy)$ equals

$$\omega(w^{-1}xywv_{ic}) = \omega(ywv_{ic}\alpha_{ic}(w^{-1})\alpha_{ic}(x)).$$

Since $\alpha'_{ic}(x) = wv_{ic}\alpha_{ic}(w^{-1})\alpha_{ic}(x)\alpha_{ic}(w)v_{ic}^{-1}w^{-1}$, this equals

$$\omega(y\alpha'_{ic}(x)wv_{ic}\alpha_{ic}(w^{-1})) = \omega(w^{-1}y\alpha'_{ic}(x)wv_{ic})$$

which is $\omega'(y\alpha'_{ic}(x))$.

**Definition 5.9** *Let $\alpha$ be a flow on a unital $C^*$-algebra. We define $K^\alpha \subset \mathbf{R} \times A^*$ by*

$$K^\alpha = \{(c, \omega) \mid \omega \text{ is a KMS functional at } c\},$$

*where KMS functional means KMS state multiplied by a non-negative constant. Then $K$ is a closed subset of $\mathbf{R} \times A^*$ and each section at $c \in \mathbf{R}$ is a lattice. We call $K^\alpha$ the KMS field for $\alpha$.*

**Proposition 5.10** *Let $u$ be an $\alpha$-cocycle. Then the KMS fields for $\alpha$ and for $\mathrm{Ad}\, u\alpha$ are isomorphic.*

*Proof.* When $u_t = wv_t\alpha(w^{-1})$ as in the previous lemma, the desired map is given by

$$\omega \mapsto \omega(w^{-1} \cdot wv_{ic}).$$

## 6 Ideals of $A \times_\alpha \mathbf{R}$

Let $\alpha$ be a flow on a unital C*-algebra. The dual flow $\hat{\alpha}$ is a flow on the crossed product $A \times_\alpha \mathbf{R}$. By the Takesaki-Takai duality $(A \times_\alpha \mathbf{R}, \hat{\alpha})$ is a complete invariant for the cocycle perturbations of $\alpha$.

If $\alpha$ has a ground state then it induces a covariant representation $(\pi, U_t = e^{itH})$ such that $H \geq 0$. Then the representation $\pi \times U$ of the crossed product $A \times_\alpha \mathbf{R}$ is not faithful. Since

$$(\pi \times U)(\lambda(f)) = \int e^{itH} f(t)dt = \hat{f}(-H),$$

the kernel contains $\lambda(f)$, $f \in K^1(\mathbf{R})$ with $\mathrm{supp}(\hat{f}) \subset (0, \infty)$. If $I$ is the ideal of $A \times_\alpha \mathbf{R}$ generated by such $\lambda(f)$, then $t \mapsto \hat{\alpha}_t(I)$ is decreasing and $I$ satisfies

$$\overline{\bigcup_t \hat{\alpha}_t(I)} = A \times_\alpha \mathbf{R}$$

and

$$\bigcap_t \hat{\alpha}_t(I) = \{0\}.$$

If $\alpha$ has a ceiling state then $A \times_\alpha \mathbf{R}$ has an ideal $J$ such that $t \mapsto \hat{\alpha}_t(J)$ is increasing from $\{0\}$ to $A \times_\alpha \mathbf{R}$.

For $\lambda > 0$ and an $\alpha$-invariant hereditary C*-subalgebra $B$ of $A$ we denote by $B_\lambda$ the C*-subalgebra of $B$ generated by $B^\alpha(-\lambda, \lambda)$, where $\alpha$ also denotes the restriction of $\alpha$ to $B$. Note that $\lambda \mapsto B_\lambda$ is increasing, where $B^\alpha(U)$ is the closure of $\{x \in B \mid \mathrm{Sp}_\alpha(x) \subset U\}$ for an open set $U$.

**Definition 6.1** *We say that $\alpha$ satisfies the* **no energy gap condition** *if the following holds: $B_\lambda = B$ for any $\lambda > 0$ and for any $\alpha$-invariant hereditary C*-subalgebra $B$ of $A$.*

Let $\alpha$ be a UHF flow on $A = \bigotimes_n M_2$ of the form

$$\alpha_t = \bigotimes_n \begin{pmatrix} e^{it\lambda_n} & 0 \\ 0 & 1 \end{pmatrix}.$$

If $\lambda_n \longrightarrow 0$ and $\sum_n |\lambda_n| = \infty$ then $\alpha$ satisfies the no energy gap condition.

**Proposition 6.2** *Suppose that $\lambda_n \longrightarrow 0$ and $\sum_n |\lambda_n|^2 = \infty$ in the above description of $\alpha$. If $\beta$ is a flow on $B$ then the flow $\alpha \otimes \beta$ on $A \otimes B$ satisfies the no energy gap condition.*

**Theorem 6.3** *Let $\alpha$ be a flow on a $C^*$-algebra $A$. Suppose that for each $t \neq 0$ $A$ is $\alpha_t$-simple and $\mathbf{T}(\alpha_t) = \mathbf{T}$. Then the following conditions are equivalent:*

1. *$\alpha$ satisfies the no energy gap condition.*
2. *All primitive ideals of $A \times_\alpha \mathbf{R}$ are monotone under $\hat{\alpha}$.*
3. *For any $B \in H^\alpha(A)$ and for any inner perturbation of $\beta$ of $\alpha|B$, $B_{(0,\lambda)}$ is independent of $\lambda > 0$ and $B_{(-\lambda,0)}$ is independent of $\lambda > 0$, where*

$$B_V = \overline{B^\beta(V)^* B B^\beta(V)}$$

*for any open subset $V$ of $\mathbf{R}$.*

*Moreover if the above conditions are satisfied, then $\mathbf{R}_C(\alpha) = \mathbf{R}$ (or $A \times_\alpha \mathbf{R}$ is prime).*

## 7 Traces on $A \times_\alpha \mathbf{R}$

**Definition 7.1** *Let $B$ be a non-unital $C^*$-algebra. A trace on $B$ is a function $\tau : B_+ \to [0, \infty]$ such that*

1. $\tau(\gamma x) = \alpha\tau(x), \quad x \in B_+, \ \gamma \in \mathbf{R}_+;$
2. $\tau(x + y) = \tau(x) + \tau(y), \quad x, y \in B_+;$
3. $\tau(u^* x u) = \tau(x), \quad x \in B, \ u \in \tilde{B}.$

*We say that $\tau$ is* **densely-defined** *if $B_+^\tau = \{x \in B_+ \mid \tau(x) < \infty\}$ is dense in $B_+$ and that $\tau$ is* **lower semi-continuous** *if $\{x \in B_+ \mid \tau(x) \leq \gamma\}$ is closed for every $\gamma \in \mathbf{R}_+$.*

*We call $\tau$* **minimal** *if for any $x \in B_+ \setminus B_+^\tau$ and an approximate identity $(e_i)$ in $B_+^\tau$ for the ideal obtained as the closed linear span of $B_+^\tau$ the net $\tau(x^{1/2} e_i x^{1/2})$ diverges to infinity.*

*Note that a lower semi-continuous densely-defined trace is minimal.*

Let $I$ be an ideal of $B$ and let $\phi$ be a lower semi-continuous densely-defined trace on $I$. Then one defines $\bar{\phi} : A_+ \to [0, \infty]$ by $\bar{\phi}(x) = \sup \phi(x^{1/2} e x^{1/2})$ where $e$ runs over $\{e \in I_+^\tau \mid \|e\| \leq 1\}$. Then $\bar{\phi}$ is a minimal lower semi-continuous trace on $B$.

We impose the following condition on $\alpha$:

**Definition 7.2** *We call $\alpha$* **uniformly profound** *if for each $p \in \mathbf{R}$ there is a sequence $(x_n)$ in $A$ such that $\|x_n\| = 1$, $\|[x_n, y]\| \to 0$ for $y \in A$,*

$$\mathrm{Sp}_\alpha(x_n) \subset (p - 1/n, p + 1/n),$$

*and*

$$x_n^* x_n + x_n x_n^* > 1/2.$$

The above condition on $\alpha$ is much stronger than profoundness (and $\mathbf{R}_C(\mathbf{R}) = \mathbf{R}$). If $\phi$ is a KMS state then the above condition implies that $\pi_\phi(A)$" is of type III.

For a sequence $(\lambda_n)$ in $\mathbf{R}$ let $\alpha$ be the UHF flow on $M_{2^\infty}$ given by the infinite tensor product of $\mathrm{Ad}(e^{i\lambda_n t} \oplus 1)$. Suppose that $\lambda_n \to 0$. Then

$$\sum_n \lambda_n^2 = \infty$$

if and only if $\alpha$ is uniformly profound. The tensor product $\alpha \otimes \beta$ with any flow $\beta$ is also uniformly profound.

We denote by $A \times_\alpha \mathbf{R}$ the crossed product of $A$ by $\alpha$. The canonical unitary multiplier flow of $A \times_\alpha \mathbf{R}$, denoted by $\lambda_t$, $t \in \mathbf{R}$, satisfies that

$$\lambda_t a = \alpha_t(a)\lambda_t, \ a \in A.$$

Recall $K^1(\mathbf{R}) = \{f \in L^1(\mathbf{R}) \mid \mathrm{supp}(\hat{f}) \text{ is compact}\}$. For $f \in K^1(\mathbf{R})$ we write $\lambda(f) = \int f(t)\lambda_t dt$.

**Lemma 7.3** *Suppose that $A$ is unital and that $\alpha$ is uniformly profound. Let $\tau$ be a non-zero lower semi-continuous densely-defined trace on $A \times_\alpha \mathbf{R}$ such that the GNS representation $\pi_\tau$ is factorial. Then there are $c \in \mathbf{R}$ and $C > 0$ and a KMS state $\omega$ on $A$ at $c$ such that*

$$\tau(a\lambda(f)) = C\omega(a) \int \hat{f}(q)e^{-cq}dq$$

*for $f \in K^1(R)$. Moreover it follows that $\tau\hat{\alpha}_p = e^{-cp}\tau$ for $p \in \mathbf{R}$.*

*Proof.* Since $\tau$ is well-defined on $\lambda(f)$, $f \in K^1(\mathbf{R})$, there is a Radon measure $\mu$ on $\mathbf{R}$ such that

$$\tau(\lambda(f)) = \int \hat{f}(q)d\mu(q), \ \ f \in K^1(\mathbf{R}).$$

Let $(x_n)$ be a sequence in $A$ for $p \in \mathbf{R}$ as in the definition of uniform profoundness. Since $\|\lambda_t x_n - e^{ipt}x_n\lambda_t\| \to 0$, it follows that for any $f \in K^1(\mathbf{R})$

$$\|\lambda(f)x_n - x_n\lambda(\chi_p f)\| \to 0,$$

where $\chi_p(t) = e^{ipt}$. Let $z_n = \lambda(f)x_n x_n^* \lambda(f) - x_n\lambda(\chi_p f)\lambda(\chi_p f)^* x_n^*$, which converges to zero in norm. If $g \in K^1(\mathbf{R})$ is such that

$\hat{g} \geq 0$ and $\hat{g}$ is 1 on a neighborhood of $\mathrm{supp}(\hat{f})$, it follows that $\lambda(g)z_n = z_n$. Since $\tau(\lambda(g)) < \infty$, we conclude that $\tau(z_n) \to 0$, i.e., $\tau(\lambda(f)x_n x_n^* \lambda(f)^*) - \tau(\lambda(\chi_p f)^* x_n^* x_n \lambda(\chi_p f)) \to 0$. Since $\pi_\tau$ is factorial and $(x_n x_n^*)$ and $(x_n^* x_n)$ approximately commute with all elements of $A \times_\alpha \mathbf{R}$, we may suppose that $\pi_\tau(x_n x_n^*) \to c_1 1$ and $\pi_\tau(x_n^* x_n) \to c_2 1$ weakly. Thus we can conclude that $c_1 \tau(\lambda(f)\lambda(f)^*) = c_2 \tau(\lambda(\chi_p f)^* \lambda(\chi_p f))$. Since $c_1 + c_2 \geq 1/2$, we deduce that $c_i > 0$. (If $c_1 = 0$ then $\tau(\lambda(\chi_p f)^* \lambda(\chi_p f)) = 0$ for all $f \in K^1(\mathbf{R})$, i.e., $\tau = 0$.) Set $a_p = c_1/c_2$; then it follows that $d\mu(\cdot + p) = a_p d\mu$. Since $a_p$ is continuous in $p$ and $a_p a_q = a_{p+q}$ for $p, q \in \mathbf{R}$ one can conclude that $a_p = e^{-cp}$ for some $c \in \mathbf{R}$. Since $e^{cq} d\mu(q)$ is translation-invariant, one can conclude that $d\mu(q) = Ce^{-cq} dq$.

Let $f \in K^1(\mathbf{R})$ be such that $\lambda(f) \geq 0$ and define a state $\omega_f$ on $A$ by $\omega(a) = \tau(a\lambda(f))/\tau(\lambda(f))$. Using the sequence $(x_n)$ given above, we conclude that

$$\frac{\tau(x_n x_n^* a \lambda(f))}{c_1 \tau(\lambda(f))} \to \omega_f(a).$$

But the left hand side also converges to $\omega_{\chi_p f}(a)$ as follows by computing $\tau(x_n^* a \lambda(f) x_n)$. Thus one can conclude that $\omega_f = \omega_{\chi_p f}$ for all $p \in \mathbf{R}$. In this way argue that $\omega_f$ is independent of $f$ and then that it is a KMS state at $c$.

**Remark 7.4** *We need some condition on $\alpha$ to obtain the conclusion in the above lemma. If $\alpha_t = \mathrm{id}$ then $A \times_\alpha \mathbf{R} \cong A \otimes C_0(\mathbf{R})$ and it has many tracial states if $A$ has. If $\alpha$ is the UHF flow on $A = M_{2^\infty}$ determined by a sequence $(\lambda_n)$ such that $\lambda_n \to 0$, $\sum_n |\lambda_n| = \infty$, and $\sum_n \lambda_n^2 < \infty$, then $\alpha$ is profound and $A \times_\alpha \mathbf{R}$ is prime and has tracial states.*

Suppose that there is a lower semi-continuous densely-defined trace $\tau_c$ for each $c > 0$ such that

$$\tau_c(\lambda(f)) = \int e^{-cq} \hat{f}(q) dq.$$

Then by taking the limit $c \to \infty$ we obtain a trace $\tau : (A \times_\alpha \mathbf{R})_+ \to [0, \infty]$ such that $\tau(\lambda(f)) = 0$ for positive $f \in K^1(\mathbf{R})$ with $\mathrm{supp}\hat{f} \subset (0, \infty)$ and $\tau(\lambda(f)) = \infty$ for positive $f \in K^1(\mathbf{R})$ with $\mathrm{supp}\hat{f} \subset (-\infty, 0)$. Hence $\{x \in (A \times_\alpha \mathbf{R})_+ \mid \tau(x) = 0\}$ is a non-zero hereditary cone invariant under the inner automorphisms. Thus the linear span is a proper ideal. One can conclude that it cannot be dense and its closure is also a proper ideal. This is of course well-known from the existence of ground states.

**Lemma 7.5** *Suppose that $A$ is simple and unital and suppose that $\alpha$ is uniformly profound. Then any minimal lower semi-continuous trace on $A \times_\alpha \mathbf{R}$ is densely-defined.*

Let $T(A \times_\alpha \mathbf{R})$ denote the set of lower semi-continuous densely-defined traces on $A \times_\alpha \mathbf{R}$.

Since the Pedersen ideal $P$ is the smallest dense ideal of $A \times_\alpha \mathbf{R}$, $\tau$ is well-defined on $P$ for all $\tau \in T = T(A \times_\alpha \mathbf{R})$. Since $\tau$ is determined by $\tau | P$ we may regard $T$ as a convex cone. We equip $T$ with the topology determined by $\tau \mapsto \tau(x)$, $x \in P$, which is equivalent to the one determined by $\tau \mapsto \tau(a\lambda(g))$, $a \in A, g \in K^1(\mathbf{R})$.

**Proposition 7.6** *Let $\alpha$ be a uniformly profound flow on a unital $C^*$-algebra $A$ and suppose that there is one and only one KMS state with respect to $\alpha$ at each $c \in \mathbf{R}$. Then the convex cone $T(A \times_\alpha \mathbf{R})$ is isomorphic to the convex cone $\mathcal{M}$ of finite measures $\mu$ on $\mathbf{R}$ satisfying $\int e^{-ps} d\mu(s) < \infty$ for all $p \in \mathbf{R}$ with the topology defined by $\mu \mapsto \int e^{-ps} d\mu(s)$, $p \in \mathbf{R}$.*

## 8 Problems

1. When $\alpha$ is a flow on an AF algebra (or a UHF algebra) clarify the relations among the four conditions on $\alpha$; asymptotically inner, approximately inner, quasi-diagonal, and pseudo-diagonal.

2. Give a necessary and sufficient condition for a quantum spin flow to be an approximate AF flow.

3. Probably there are many flows $\alpha$ on $A = M_{2^\infty}$ such that $T(A \times_\alpha \mathbf{R}) \cong \mathcal{M}$ and the primitive ideal space is $\{0\} \sqcup \mathbf{R} \sqcup \mathbf{R}$ (where one of $\mathbf{R}$ represents an increasing ideal under $\hat{\alpha}$ the other a decreasing). Are there many $A \times_\alpha \mathbf{R}$?

4. Under the previous situation if there is another flow $\beta$ which behaves in the same way as $\hat{\alpha}$ on the primitive ideals, is $\beta$ a cocycle perturbation of $\hat{\alpha}$?

5. Under the previous situation if there is a flow $\beta$ which increases one primitive ideal and decreases anther, how closely is $\beta$ related to $\hat{\alpha}$?

6. If $\alpha$ is a flow on the UHF algebra $M_{2^\infty}$ commuting with the gauge action $\gamma$ of $\mathbf{T}$, can we conclude that $\alpha$ is quasi-diagonal or approximately inner? Here $\gamma$ is given by
$$\gamma_z = \bigotimes \mathrm{Ad} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}.$$

7. For any $\epsilon > 0$ is there a $\delta > 0$ satisfying the following condition? If $\alpha$ is a flow on a unital C*-algebra $A$ such that $A \supset B \ni 1_A$ and $B \cong M_n$ for some $n$ and if $\sup_{t\in[0,1]} \mathrm{dist}(M_n, \alpha_t(M_n)) < \delta$ then there is an $\alpha$-cocycle $u$ such that $\mathrm{Ad}\, u_t\alpha_t(M_n) = M_n$ and $\sup_{t\in[0,1]} \|u_t - 1\| < \epsilon$.

8. Let $\alpha$ be a flow on a Cuntz algebra $A$. Prove that $\alpha$ has the Rohlin property if $A \times_\alpha \mathbf{R}$ is purely infinite.

9. Prove that Rohlin flows on the Cuntz algebra are cocycle conjugate.

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